# Grothendieck ring and Verlinde-like formula for the $\mathcal{W}$-extended logarithmic minimal model $\mathcal{WLM}(1, p)$



2010 J. Phys. A: Math. Theor. 43 045211

(http://iopscience.iop.org/1751-8121/43/4/045211)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **43** (2010) 045211 (13pp) doi:10.1088/1751-8113/43/4/045211

# Grothendieck ring and Verlinde-like formula for the $\mathcal{W}$-extended logarithmic minimal model $\mathcal{WLM}(1, p)$

**Paul A Pearce[1], Jørgen Rasmussen[1] and Philippe Ruelle[2]**

[1] Department of Mathematics and Statistics, University of Melbourne, Parkville, Victoria 3010, Australia
[2] Institut de Physique Théorique, Université catholique de Louvain, 1348 Louvain-la Neuve, Belgium

E-mail: P.Pearce@ms.unimelb.edu.au, j.rasmussen@ms.unimelb.edu.au and philippe.ruelle@uclouvain.be

Received 4 August 2009, in final form 6 November 2009
Published 8 January 2010
Online at stacks.iop.org/JPhysA/43/045211

**Abstract**
We consider the Grothendieck ring of the fusion algebra of the $\mathcal{W}$-extended logarithmic minimal model $\mathcal{WLM}(1, p)$. Informally, this is the fusion ring of $\mathcal{W}$-irreducible characters so it is blind to the Jordan block structures associated with reducible yet indecomposable representations. As in the rational models, the Grothendieck ring is described by a simple graph fusion algebra. The $2p$-dimensional matrices of the regular representation are mutually commuting but not diagonalizable. They are brought simultaneously to Jordan form by the modular data coming from the full $(3p - 1)$-dimensional $S$-matrix which includes transformations of the $p - 1$ pseudo-characters. The spectral decomposition yields a Verlinde-like formula that is manifestly independent of the modular parameter $\tau$ but is, in fact, equivalent to the Verlinde-like formula recently proposed by Gaberdiel and Runkel involving a $\tau$-dependent $S$-matrix.

PACS number: 11.25.Hf

## 1. Introduction

Fusion algebras carry fundamental and physically important information about the structure of conformal field theories (CFTs). In their most succinct form, fusion rules are encoded in graph fusion algebras [1]. For rational CFTs, the celebrated Verlinde formula [2] states that the *modular S-matrix diagonalizes the fusion rules*. More precisely, the modular $S$-matrix is the similarity matrix which simultaneously brings the graph fusion matrices (in the regular representation) into diagonal form. Recall that, in the rational setting, the modular $S$-matrix specifies the linear transformations of the finite number of irreducible characters under the transformation $\tau \to -1/\tau$. The properties of modular invariance [3] on the torus

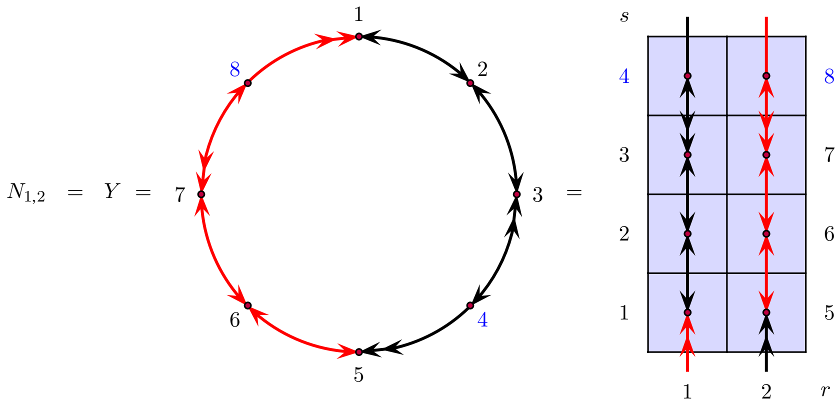


**Figure 1.** The fundamental fusion graph $Y = N_{1,2}$ of the Grothendieck ring of $\mathcal{WLM}(1,4)$ and the same graph superimposed on the Kac table. The labels are $1 = (1,1)$, $2 = (1,2)$, $3 = (1,3)$, $4 = (1,4)$, $5 = (2,1)$, $6 = (2,2)$, $7 = (2,3)$ and $8 = (2,4)$. The fundamental fusion graphs for larger values of $p$ are obtained by adding additional doubly directed bonds at the positions labeled 1 and 5.

(This figure is in colour only in the electronic version)

and modular covariance [4] in the presence of a boundary subsequently played a major role in the understanding of rational CFTs. In particular, it led to the Cardy formula for the cylinder partition functions in terms of the modular $S$-matrix thus establishing the boundary operator content of rational CFTs.

The Verlinde and Cardy formulas [1, 5] play a central role in the general formulation of rational CFTs. However, the CFTs describing systems such as polymers and critical percolation, are not rational CFTs—they are logarithmic CFTs. With the recent upsurge of interest in logarithmic CFTs [6, 7], a natural question concerns the generalized form of the Verlinde and Cardy formulas in the context of logarithmic CFTs. This poses a number of serious challenges: (i) the appearance of indecomposable representations whose characters are linear combinations of irreducible characters, (ii) the fact that the irreducible characters transform under $\tau \to -1/\tau$ through a $\tau$-dependent $S$-matrix and (iii) the non-diagonalizability of fusion matrices. A more detailed account of these difficulties can be found in [8]. The Verlinde and Cardy formulas are concerned with spectra and are blind to the Jordan block structures associated with indecomposability which is a characteristic of logarithmic theories. For this reason, it suffices to work with the Grothendieck ring rather than the full fusion ring of representations. In the cases of interest here, the Grothendieck ring is the quotient fusion algebra obtained by elevating the various character identities to equivalence relations between the corresponding representations. Informally, it is the *fusion ring of irreducible characters*. While this effectively solves problem (i), the other two problems need to be resolved.

In this paper, we consider logarithmic minimal models [9, 10] in the $\mathcal{W}$-extended picture [11, 12], namely the models $\mathcal{WLM}(1, p)$ for integer $p > 1$. The case $p = 2$ is symplectic fermions [13]. For these models, we present a derivation of a Verlinde-like formula close in spirit to the familiar treatment of the Verlinde formula in the rational setting which is based on spectral decompositions. The central charge of the $\mathcal{WLM}(1, p)$ model is $c = 13 - 6\big(p + \frac{1}{p}\big)$ and the chiral vertex algebra $\mathcal{W}_p$ is generated by three fields of dimension $2p - 1$, with $2p$ irreducible representations. The associated $2p$-dimensional Grothendieck fusion matrices are, in general, not diagonalizable and typically contain Jordan blocks of rank 2. We show that a single similarity matrix $Q$ simultaneously brings these matrices to Jordan form, albeit not necessarily strict Jordan *canonical* form. The matrix $Q$ is simply related to the modular matrix $S$. Specifically, the independent columns of the $S$-matrix provide a system of eigenvectors and generalized eigenvectors which form the columns of $Q$. Moreover, the eigenvalues of the Grothendieck fusion matrices are simple fractions in the entries of the $S$-matrix. It is in this sense, that our Verlinde-like formula reduces to a spectral decomposition of the Grothendieck fusion matrices, for which the modular $S$-matrix provides the eigendata. We stress, though, that the matrix $Q$ and its realization in terms of modular data are far from unique.

A complicating factor when studying $\mathcal{WLM}(1, p)$ is that, strictly speaking, the set of irreducible characters does not close under the modular transformation $\tau \to -1/\tau$. One can work with a $2p$-dimensional but $\tau$-dependent $S$-matrix (it is linear in $\tau$), or one can alternatively enlarge the space of characters by introducing $p - 1$ so-called pseudo-characters [14–17] thereby obtaining a proper representation of the modular group. In either formulation, the $S$-matrices contain equivalent modular data. We work with the proper $S$-matrix but present our final result for the Verlinde-like formula in both formulations and so establish the equivalence of our Verlinde-like formula to the formula obtained recently by Gaberdiel and Runkel [18]. Although their form of the Verlinde-like formula involves $\tau$, it was checked numerically to be independent of $\tau$ and to correctly reproduce the Grothendieck ring structure constants. We confirm this analytically and show that our Verlinde-like formula reproduces the $\tau$-independent part of their formula thereby completing the proof of their formula.

Other approaches to a Verlinde-like formula for the $\mathcal{WLM}(1, p)$ models have also been proposed. In [8], the problem (ii) is circumvented by introducing automorphy factors in order to obtain a true linear representation of the modular group. The matrix representing the transformation $\tau \to -1/\tau$ is shown to block-diagonalize the Grothendieck fusion matrices, the block-diagonal form being expressible in terms of the entries of this same matrix. Albeit different from our formula, the resulting formula correctly reproduces the fusion coefficients (in the Grothendieck ring). As in [18], the modular matrix is, in principle, $\tau$-dependent but the working requires the specific choice $\tau = i$. This block-diagonalization similarly misses the Jordan form which we believe is key to a proper interpretation of a Verlinde-like formula in a logarithmic setting. Specifically, the modular $S$-matrix enters by providing the eigenvectors and generalized eigenvectors that form the columns of the similarity matrix bringing the fusion matrices to Jordan form. The article [19] generalizes the approach of [8] to include the indecomposable representations and proposes an equivalent limit-Verlinde-like formula. In a separate development [20], a Verlinde-like formula based on the *union* of irreducible characters and pseudo-characters has been derived.

Finally, in a companion paper [21], our present study of the Grothendieck ring of $\mathcal{WLM}(1, p)$ is extended to the fusion algebra itself. The analysis is very similar though technically more involved due to the presence of Jordan blocks of rank 3 in the spectral decompositions of the fusion matrices. This also affects the form of the resulting Verlinde-like formulas.

## 2. The logarithmic minimal model $\mathcal{WLM}(1, p)$

A $\mathcal{W}$-extended logarithmic minimal model $\mathcal{WLM}(1, p)$ is defined [11, 12] for every positive integer $p > 1$. It has central charge

$$c = 13 - 6\left(p + \frac{1}{p}\right) \tag{2.1}$$

while the conformal weights are given by

$$\Delta_{r,s} = \frac{(rp-s)^2-(p-1)^2}{4p}, \qquad r, s \in \mathbb{N}. \tag{2.2}$$

There are $2p$ irreducible representations

$$\{(r,s)_{\mathcal{W}};\ r \in \mathbb{Z}_{1,2}, s \in \mathbb{Z}_{1,p}\}, \qquad \Delta[(r,s)_{\mathcal{W}}] = \Delta_{r,s} \tag{2.3}$$

and $2p-2$ indecomposable rank-2 representations

$$\big\{\big(\mathcal{R}_r^b\big)_{\mathcal{W}};\ r \in \mathbb{Z}_{1,2}, b \in \mathbb{Z}_{1,p-1}\big\} \tag{2.4}$$

where we have introduced the notation $\mathbb{Z}_{n,m} = \mathbb{Z} \cap [n,m]$. The irreducible characters are given by

$$\hat{\chi}_{r,s}(q) = \chi[(r,s)_{\mathcal{W}}](q) = \frac{1}{\eta(q)} \sum_{j\in\mathbb{Z}} (2j+r) q^{p(j+\frac{rp-s}{2p})^2} \tag{2.5}$$

where $\eta(q)$ is the Dedekind eta function:

$$\eta(q) = q^{1/24} \prod_{m=1}^{\infty} (1-q^m). \tag{2.6}$$

The characters of the indecomposable rank-2 representations are given by

$$\chi\big[\big(\mathcal{R}_1^b\big)_{\mathcal{W}}\big](q) = \chi\big[\big(\mathcal{R}_2^{p-b}\big)_{\mathcal{W}}\big](q) = 2\hat{\chi}_{2,b}(q) + 2\hat{\chi}_{1,p-b}(q). \tag{2.7}$$

There are $p+1$ so-called projective characters. These are the characters $\hat{\chi}_{r,p}(q)$ and (2.7) of the projective representations:

$$\big\{(r,p)_{\mathcal{W}}, \big(\mathcal{R}_r^b\big)_{\mathcal{W}};\ r \in \mathbb{Z}_{1,2}, b \in \mathbb{Z}_{1,p-1}\big\}. \tag{2.8}$$

### 2.1. *Fusion algebra*

The fundamental fusion algebra

$$\mathrm{Fund}[\mathcal{WLM}(1,p)] = \big\langle (r,s)_{\mathcal{W}}, \big(\mathcal{R}_r^b\big)_{\mathcal{W}};\ r \in \mathbb{Z}_{1,2}, s \in \mathbb{Z}_{1,p}, b \in \mathbb{Z}_{1,p-1}\big\rangle \tag{2.9}$$

of $\mathcal{WLM}(1,p)$ is commutative and associative. The underlying fusion rules read [11, 18, 22, 23]

$$\begin{aligned}
(r,s)_{\mathcal{W}} \otimes (r',s')_{\mathcal{W}} &= \bigoplus_{j=|s-s'|+1,\,\mathrm{by}\,2}^{p-|p-s-s'|-1} (r\cdot r', j)_{\mathcal{W}} \oplus \bigoplus_{\beta=\epsilon(s+s'-p-1),\,\mathrm{by}\,2}^{s+s'-p-1} \big(\mathcal{R}_{r\cdot r'}^{\beta}\big)_{\mathcal{W}} \\
(r,s)_{\mathcal{W}} \otimes \big(\mathcal{R}_{r'}^{b}\big)_{\mathcal{W}} &= \bigoplus_{\beta=|s-b|+1,\,\mathrm{by}\,2}^{p-|p-s-b|-1} \big(\mathcal{R}_{r\cdot r'}^{\beta}\big)_{\mathcal{W}} \oplus \bigoplus_{\beta=\epsilon(s-b-1),\,\mathrm{by}\,2}^{s-b-1} 2\big(\mathcal{R}_{r\cdot r'}^{\beta}\big)_{\mathcal{W}} \\
&\quad \oplus \bigoplus_{\beta=\epsilon(s+b-p-1),\,\mathrm{by}\,2}^{s+b-p-1} 2\big(\mathcal{R}_{2\cdot r\cdot r'}^{\beta}\big)_{\mathcal{W}} \\
\big(\mathcal{R}_r^b\big)_{\mathcal{W}} \otimes \big(\mathcal{R}_{r'}^{b'}\big)_{\mathcal{W}} &= \bigoplus_{\beta=\epsilon(p-b-b'-1),\,\mathrm{by}\,2}^{p-|b-b'|-1} 2\big(\mathcal{R}_{r\cdot r'}^{\beta}\big)_{\mathcal{W}} \oplus \bigoplus_{\beta=\epsilon(p-b-b'-1),\,\mathrm{by}\,2}^{|p-b-b'|-1} 2\big(\mathcal{R}_{r\cdot r'}^{\beta}\big)_{\mathcal{W}} \\
&\quad \oplus \bigoplus_{\beta=\epsilon(b+b'-1),\,\mathrm{by}\,2}^{p-|p-b-b'|-1} 2\big(\mathcal{R}_{2\cdot r\cdot r'}^{\beta}\big)_{\mathcal{W}} \oplus \bigoplus_{\beta=\epsilon(b+b'-1),\,\mathrm{by}\,2}^{|b-b'|-1} 2\big(\mathcal{R}_{2\cdot r\cdot r'}^{\beta}\big)_{\mathcal{W}}
\end{aligned} \tag{2.10}$$

where we have introduced $\big(\mathcal{R}_r^0\big)_{\mathcal{W}} \equiv (r,p)_{\mathcal{W}}$ and

$$\epsilon(n) = \frac{1-(-1)^n}{2}, \qquad n\cdot m = \frac{3-(-1)^{n+m}}{2}, \qquad n,m \in \mathbb{Z}. \tag{2.11}$$

### *2.2. Fusion matrices and polynomial fusion ring*

We let

$$\left\{N_{(r,s)_{\mathcal{W}}}, N_{(\mathcal{R}_r^b)_{\mathcal{W}}};\ r\in\mathbb{Z}_{1,2}, s\in\mathbb{Z}_{1,p}, b\in\mathbb{Z}_{1,p-1}\right\} \tag{2.12}$$

denote the set of fusion matrices realizing the fusion algebra (2.10) of $\mathcal{WLM}(1,p)$. These are all $(4p-2)$-dimensional square matrices. Special notation is introduced for the two fundamental fusion matrices

$$X=N_{(2,1)_{\mathcal{W}}},\qquad Y=N_{(1,2)_{\mathcal{W}}}. \tag{2.13}$$

From [24], we have

$$N_{(r,s)_{\mathcal{W}}}=X^{r-1}U_{s-1}\left(\frac{Y}{2}\right),\qquad N_{(\mathcal{R}_r^b)_{\mathcal{W}}}=2X^{r-1}T_b\left(\frac{Y}{2}\right)U_{p-1}\left(\frac{Y}{2}\right) \tag{2.14}$$

where $T_n$ and $U_n$ are Chebyshev polynomials of the first and second kind, respectively. It follows from the refinement in [21] of the discussion of the associated quotient polynomial fusion ring in [24] that

$$\mathrm{Fund}[\mathcal{WLM}(1,p)]\simeq\mathbb{C}[X,Y]/(X^2-1,\tilde{P}_{1,p}(X,Y)) \tag{2.15}$$

where

$$\tilde{P}_{1,p}(X,Y)=\left(X-T_p\left(\frac{Y}{2}\right)\right)U_{p-1}\left(\frac{Y}{2}\right). \tag{2.16}$$

### *2.3. Modular S-matrix*

The set of irreducible characters (2.5) does not close under modular transformations. Instead, a representation of the modular group is obtained [17] by enlarging this set with the $p-1$ *pseudo-characters*

$$\hat{\chi}_{0,b}(q)=\mathrm{i}\tau(b\hat{\chi}_{1,p-b}(q)-(p-b)\hat{\chi}_{2,b}(q)) \tag{2.17}$$

where the modular parameter is

$$q=\mathrm{e}^{2\pi\mathrm{i}\tau}. \tag{2.18}$$

Writing the associated modular $S$-matrix in block form with respect to the distinction between proper characters $\hat{\chi}_{r,s}(q)$ and pseudo-characters $\hat{\chi}_{0,b}(q)$, the matrix is

$$S=\begin{pmatrix}S_{r,s}^{r',s'} & S_{r,s}^{0,b'}\\ S_{0,b}^{r',s'} & S_{0,b}^{0,b'}\end{pmatrix}=\begin{pmatrix}\frac{(2-\delta_{s',p})(-1)^{rs'+r's+rr'p}s\cos\frac{ss'\pi}{p}}{p\sqrt{2p}} & \frac{2(-1)^{rb'}\sin\frac{sb'\pi}{p}}{p\sqrt{2p}}\\ \frac{2(-1)^{r'b}(p-s')\sin\frac{bs'\pi}{p}}{\sqrt{2p}} & 0\end{pmatrix}. \tag{2.19}$$

This matrix is not symmetric and not unitary but satisfies $S^2=I$. We note that

$$S_{r,s}^{1,p-b}=S_{r,s}^{2,b} \tag{2.20}$$

implying that, under the modular transformation $\tau\to\frac{-1}{\tau}$, the $2p$ irreducible characters transform into linear combinations of the $p+1$ projective characters (with expansion coefficients $S_{r,s}^{r',p}$ and $\frac{1}{2}S_{r,s}^{2,b}$) and the $p-1$ pseudo-characters (with expansion coefficients $S_{r,s}^{0,b}$), only. We also note that, formally,

$$S_{r,s}^{2,b}=\frac{\partial}{\partial\theta_b}S_{r,s}^{0,b},\qquad \theta_b=\frac{b\pi}{p}. \tag{2.21}$$

Alternatively, one can introduce the $2p$-dimensional, $\tau$-dependent (and thus *improper*) $S$-matrix

$$\mathcal{S} - \mathrm{i}\tau\tilde{\mathcal{S}} \tag{2.22}$$

(here written in calligraphic to distinguish it from the proper $S$-matrix in (2.19)) obtained by expanding the pseudo-characters in terms of the irreducible characters. Its entries thus read

$$\mathcal{S}_{r,s}^{r',s'} = S_{r,s}^{r',s'}, \qquad \tilde{\mathcal{S}}_{r,s}^{r',s'} = \frac{2(-1)^{rs'+r's+rr'p}(p-s')\sin\frac{ss'\pi}{p}}{p\sqrt{2p}} \tag{2.23}$$

from which it follows that

$$(p-b)\tilde{\mathcal{S}}_{r,s}^{1,p-b} = -b\tilde{\mathcal{S}}_{r,s}^{2,b} \tag{2.24}$$

and

$$\tilde{\mathcal{S}}_{r,s}^{1,b'} = -(p-b')S_{r,s}^{0,p-b'}, \qquad \tilde{\mathcal{S}}_{r,s}^{2,b'} = (p-b')S_{r,s}^{0,b'}, \qquad \tilde{\mathcal{S}}_{r,s}^{r',p} = 0. \tag{2.25}$$

It is easily seen that an expression can be written in terms of the proper $S$-matrix $S$ if and only if it can be written in terms of the improper $S$-matrix $\mathcal{S} - \mathrm{i}\tau\tilde{\mathcal{S}}$.

## 3. Grothendieck ring of $\mathcal{WLM}(1,p)$

The Grothendieck ring of $\mathcal{WLM}(1,p)$ is obtained by elevating the character identities of $\mathcal{WLM}(1,p)$ to relations between the corresponding generators of the fusion algebra Fund$[\mathcal{WLM}(1,p)]$ of $\mathcal{WLM}(1,p)$. From (2.7), we thus impose the equivalence relations

$$\left(\mathcal{R}_1^b\right)_{\mathcal{W}} \sim \left(\mathcal{R}_2^{p-b}\right)_{\mathcal{W}} \sim 2(2,b)_{\mathcal{W}} \oplus 2(1,p-b)_{\mathcal{W}}, \qquad b \in \mathbb{Z}_{1,p-1}. \tag{3.1}$$

Following (2.10), it is straightforwardly verified that

$$\begin{aligned}\mathrm{Grot}[\mathcal{WLM}(1,p)] &:= \langle G_{r,s};\, r \in \mathbb{Z}_{1,2},\, s \in \mathbb{Z}_{1,p-1}\rangle_{1,p} \\ &\simeq \mathrm{Fund}[\mathcal{WLM}(1,p)]/\sim\end{aligned} \tag{3.2}$$

where the equivalence relation $\sim$ is defined in (3.1) and where the rules with respect to the Grothendieck multiplication $*$ are

$$G_{r,s} * G_{r',s'} = \sum_{j=|s-s'|+1,\,\mathrm{by}\,2}^{p-|p-s-s'|-1} G_{r\cdot r',j} + \sum_{\beta=\epsilon(s+s'-p-1),\,\mathrm{by}\,2}^{s+s'-p-1} (2-\delta_{\beta,0})(G_{r\cdot r',p-\beta} + G_{2\cdot r\cdot r',\beta}). \tag{3.3}$$

Here we are using $G_{r,0} \equiv 0$. Since

$$N_{(\mathcal{R}_r^b)_{\mathcal{W}}} = 2X^{r-1}\left(T_p\left(\frac{Y}{2}\right)U_{b-1}\left(\frac{Y}{2}\right) + U_{p-b-1}\left(\frac{Y}{2}\right)\right) \tag{3.4}$$

and

$$N_{(2\cdot r,b)_{\mathcal{W}}} + N_{(r,p-b)_{\mathcal{W}}} \equiv X^{r-1}\left(XU_{b-1}\left(\frac{Y}{2}\right) + U_{p-b-1}\left(\frac{Y}{2}\right)\right) \qquad (\mathrm{mod}\ X^2-1) \tag{3.5}$$

it follows (from setting $r = b = 1$, in particular) that

$$\begin{aligned}\mathrm{Grot}[\mathcal{WLM}(1,p)] &\simeq \mathbb{C}[X,Y]\Big/\left(X^2-1,\, X - T_p\left(\frac{Y}{2}\right)\right) \\ &\simeq \mathbb{C}[Y]/\left((Y^2-4)U_{p-1}^2\left(\frac{Y}{2}\right)\right).\end{aligned} \tag{3.6}$$

With reference to (2.15), it is noted that $X - T_p\left(\frac{Y}{2}\right)$ is a divisor of $\tilde{P}_{1,p}(X, Y)$. The first isomorphism in (3.6) is given by

$$G_{2,1} \leftrightarrow X, \qquad G_{1,2} \leftrightarrow Y, \qquad G_{r,s} \leftrightarrow X^{r-1}U_{s-1}\left(\frac{Y}{2}\right) \tag{3.7}$$

while the second isomorphism is due to

$$X^2 - 1 \equiv T_p^2\left(\frac{Y}{2}\right) - 1 = \frac{1}{4}(Y^2-4)U_{p-1}^2\left(\frac{Y}{2}\right) \qquad \left(\mathrm{mod}\ X - T_p\left(\frac{Y}{2}\right)\right) \tag{3.8}$$

and allows us to write

$$G_{r,s} \leftrightarrow T_p^{r-1}\left(\frac{Y}{2}\right)U_{s-1}\left(\frac{Y}{2}\right). \tag{3.9}$$

### *3.1. Graph fusion algebra of the Grothendieck ring*

The Grothendieck ring of $\mathcal{WLM}(1, p)$ is described by a graph fusion algebra

$$N_i N_j = \sum_{k=1}^{2p} N_{ij}{}^k N_k \tag{3.10}$$

with $N_1 = N_{1,1} = I$ and $N_2 = N_{1,2} = Y$ the adjacency matrix of the fundamental graph as shown in figure 1. The fusion matrices $N_i$ are mutually commuting but in general not symmetric, not normal and not diagonalizable. Nevertheless, we will show that they can be simultaneously brought to Jordan form by a similarity transformation determined from the modular data.

The regular representation of the graph fusion algebra of the Grothendieck ring of $\mathcal{WLM}(1, p)$ is specified in terms of $2p$-dimensional matrices. In the ordered basis

$$G_{1,1}, G_{2,1}; \ldots; G_{1,s}, G_{2,s}; \ldots; G_{1,p}, G_{2,p} \tag{3.11}$$

the *fundamental* Grothendieck matrices $N_{2,1} = X$ and $N_{1,2} = Y$ are given by

$$X = C_{2p}, \qquad Y = \left(\begin{array}{cccccc|c} 0 & I & 0 & \cdots & & & 0 \\ I & 0 & I & \ddots & & & \\ 0 & I & 0 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & \\ & & & & 0 & I & 0 \\ 0 & \cdots & \cdots & 0 & I & 0 & I \\ \hline 2C & 0 & \cdots & \cdots & 0 & 2I & 0 \end{array}\right) \tag{3.12}$$

where

$$C_{2p} = \mathrm{diag}(\underbrace{C, \ldots, C}_{p}), \qquad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.13}$$

is an involutory matrix, $C_{2p}^2 = I$. The $2p$-dimensional matrix $Y$ is written here as a $p$-dimensional matrix, with $2 \times 2$ matrices $0 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ as entries, whose $p$th row and column are emphasized to indicate their special status. For $p = 2$, expression (3.12) for $Y$ is meant to reduce to

$$Y|_{p=2} = \begin{pmatrix} 0 & I \\ 2C + 2I & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{pmatrix}. \tag{3.14}$$

*3.2. Spectral decomposition*

The minimal and characteristic polynomials of $X$ are readily seen to be

$$X^2 - I = (X - I)(X + I), \qquad \det(\lambda I - X) = (\lambda - 1)^p(\lambda + 1)^p. \tag{3.15}$$

The eigenvalues of $Y$ are

$$\beta_j = 2\alpha_j = 2\cos\theta_j, \qquad \theta_j = \frac{j\pi}{p}, \qquad j \in \mathbb{Z}_{0,p}. \tag{3.16}$$

This follows from the observation, to be proven below, that the minimal and characteristic polynomials of $Y$ are

$$\begin{aligned}(Y^2 - 4I)U_{p-1}^2\left(\frac{Y}{2}\right) &= (Y - 2I)(Y + 2I)\prod_{b=1}^{p-1}(Y - \beta_b I)^2 \\ \det(\lambda I - Y) &= (\lambda - 2)(\lambda + 2)\prod_{b=1}^{p-1}(\lambda - \beta_b)^2\end{aligned} \tag{3.17}$$

where we have used that the Chebyshev polynomials of the second kind factorize as

$$U_{p-1}(x) = 2^{p-1}\prod_{b=1}^{p-1}(x - \alpha_b). \tag{3.18}$$

Expressions (3.17) imply that the Jordan canonical form $J_Y$ of $Y$ consists of $p-1$ rank-2 blocks associated with the eigenvalues $\beta_b$, $b \in \mathbb{Z}_{1,p-1}$, and a rank-1 block associated with each of the eigenvalues $\beta_0 = 2$ and $\beta_p = -2$.

## 4. Verlinde-like formulas

First, we note that the eigenvalues (3.16) can be written in terms of modular data as

$$\beta_{(r-1)p} = \frac{S_{1,2}^{r,p}}{S_{1,1}^{r,p}}, \qquad \beta_b = \frac{S_{1,2}^{0,b}}{S_{1,1}^{0,b}}. \tag{4.1}$$

Second, we wish to Jordan decompose the matrix $Y$ using a similarity matrix whose entries are given in terms of the modular data. For $r' \in \mathbb{Z}_{1,2}$ and $b \in \mathbb{Z}_{1,p-1}$, we thus define the $2p$-dimensional vectors $v_{(r'-1)p}$, $v_b$ and $w_b$ whose entries are given by

$$[v_{(r'-1)p}]_{r,s} = S_{r,s}^{r',p}, \qquad [v_b]_{r,s} = S_{r,s}^{0,b}, \qquad [w_b]_{r,s} = S_{r,s}^{2,b}. \tag{4.2}$$

The vector $v_j$, $j \in \mathbb{Z}_{0,p}$, is the unique eigenvector of $Y$ associated with the eigenvalue $\beta_j$, while the doublet $v_b$, $w_b$ forms the generalized Jordan chain

$$Y(v_b\ w_b) = (v_b\ w_b)\begin{pmatrix}\beta_b & -2\sin\theta_b \\ 0 & \beta_b\end{pmatrix}. \tag{4.3}$$

These assertions are straightforwardly proven using standard properties of the Chebyshev polynomials, such as their recurrence relations, and immediately imply the results (3.17). It follows that the similarity matrix

$$Q = \left(v_0 \mid \dots \mid v_b \quad w_b \mid \dots \mid v_p\right) \tag{4.4}$$

whose entries are given by

$$Q = \left(S_{r,s}^{1,p} \mid \dots \mid S_{r,s}^{0,b} \quad S_{r,s}^{2,b} \mid \dots \mid S_{r,s}^{2,p}\right) \tag{4.5}$$

converts the matrix realization of the Grothendieck generator $Y$ into the (in general, non-canonical) Jordan form

$$Q^{-1}YQ = \mathrm{diag}\left(\beta_0, \ldots, \begin{pmatrix} \beta_b & -2\sin\theta_b \\ 0 & \beta_b \end{pmatrix}, \ldots, \beta_p\right). \tag{4.6}$$

The entries of the inverse of the similarity matrix $Q$ are given by

$$Q^{-1} = \begin{pmatrix} S^{r,s}_{1,p} \\ \hline \vdots \\ \hline S^{r,s}_{0,b} \\ \frac{S^{1,p}_{1,p}}{S^{1,p}_{1,b}} S^{r,s}_{2,b} \\ \hline \vdots \\ \hline S^{r,s}_{2,p} \end{pmatrix}. \tag{4.7}$$

The columns are labeled by $(r,s)_{\mathcal{W}}$, while the rows are labeled by $v_0$, the $p-1$ doublets $v_b$, $w_b$ and $v_p$. As an aside, we have verified for small values of $p$ (analytically for $p = 2, 3, 4, 5, 6$ and numerically up to $p = 30$) that the determinant of $Q$ is

$$\det(Q) = \frac{(-1)^{\lfloor \frac{p-2}{2} \rfloor}}{p^{p-1}}. \tag{4.8}$$

Finally, the use of modular data obtained from transforming the pseudo-characters, as in the expression (4.7) for $Q^{-1}$, can be avoided since

$$S^{r,s}_{0,b'} = \frac{S^{1,p}_{1,p}\left(S^{1,p}_{1,p} - S^{1,p}_{1,s}\right)}{\left(S^{1,p}_{1,1}\right)^2} S^{0,b'}_{r,s}, \qquad S^{r,b}_{0,b'} = \frac{S^{1,p}_{1,p} S^{1,p}_{1,p-b}}{\left(S^{1,p}_{1,1}\right)^2} S^{0,b'}_{r,b}, \qquad S^{r,p}_{0,b'} = 0. \tag{4.9}$$

The matrices realizing the other generators can be brought to a similar form since, referring to (3.9), we have

$$N_{r,s} = f_{r,s}(Y). \tag{4.10}$$

Here, we have introduced the $2p$ functions

$$f_{r,s}(x) = T_p^{r-1}\left(\frac{x}{2}\right) U_{s-1}\left(\frac{x}{2}\right) \tag{4.11}$$

satisfying

$$f_{r,s}(\beta_0) = s, \qquad f_{r,s}(\beta_b) = \frac{(-1)^{(r-1)b}\sin s\theta_b}{\sin\theta_b}, \qquad f_{r,s}(\beta_p) = s(-1)^{(r-1)p+s-1}. \tag{4.12}$$

Using the following formula valid for any regular function $f$

$$f\begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} f(\lambda) & \mu f'(\lambda) \\ 0 & f(\lambda) \end{pmatrix} \tag{4.13}$$

we easily obtain the Jordan form for the matrices $N_{r,s}$

$$J_{r,s} = Q^{-1}N_{r,s}Q = \mathrm{diag}\left(f_{r,s}(\beta_0), \ldots, \begin{pmatrix} f_{r,s}(\beta_b) & -2\sin\theta_b f'_{r,s}(\beta_b) \\ 0 & f_{r,s}(\beta_b) \end{pmatrix}, \ldots, f_{r,s}(\beta_p)\right). \tag{4.14}$$

We emphasize that this Jordan form can be expressed as a single function acting on a Jordan *canonical* form

$$J_{r,s} = f_{r,s} \circ \phi(\mathrm{diag}(\theta_0, \mathcal{J}_{\theta_1,2}, \ldots \mathcal{J}_{\theta_b,2}, \ldots, \mathcal{J}_{\theta_{p-1},2}, \theta_p)), \qquad \phi(\theta) = 2\cos\theta \tag{4.15}$$

where

$$\mathcal{J}_{\lambda,2} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}. \tag{4.16}$$

Since the eigenvalues of the matrices $N_{r,s}$ can also be written in terms of the modular data

$$f_{r,s}(\beta_0) = \frac{S^{1,p}_{r,s}}{S^{1,p}_{1,1}}, \qquad f_{r,s}(\beta_b) = \frac{S^{0,b}_{r,s}}{S^{0,b}_{1,1}}, \qquad f_{r,s}(\beta_p) = \frac{S^{2,p}_{r,s}}{S^{2,p}_{1,1}} \tag{4.17}$$

it follows that the announced generalization of the Verlinde formula can be written as

$$N_{r,s} = Q\ \mathrm{diag}\left(\frac{S^{1,p}_{r,s}}{S^{1,p}_{1,1}}, \ldots, \begin{pmatrix} \frac{S^{0,b}_{r,s}}{S^{0,b}_{1,1}} & \frac{S^{2,b}_{r,s}}{S^{0,b}_{1,1}} - \frac{S^{2,b}_{1,1}S^{0,b}_{r,s}}{(S^{0,b}_{1,1})^2} \\ 0 & \frac{S^{0,b}_{r,s}}{S^{0,b}_{1,1}} \end{pmatrix}, \ldots, \frac{S^{2,p}_{r,s}}{S^{2,p}_{1,1}}\right) Q^{-1}. \tag{4.18}$$

In terms of the formal derivative (2.21), this formula can be expressed compactly as

$$N_{r,s} = Q\ \mathrm{diag}\left(\frac{S^{1,p}_{r,s}}{S^{1,p}_{1,1}}, \ldots, \frac{S^{0,b}_{r,s}}{S^{0,b}_{1,1}}(\mathcal{J}_{\theta_b,2}), \ldots, \frac{S^{2,p}_{r,s}}{S^{2,p}_{1,1}}\right) Q^{-1}. \tag{4.19}$$

Every entry of the matrix $N_{r,s}$ in (4.18) can be interpreted as a sum of three contributions; one obtained by summing over the projective characters, one over the pseudo-characters and one off-diagonal term with a sum over both projective and pseudo-characters. We thus have

$$[N_{r,s}]_{r',s'}^{r'',s''} = \left[N^{\mathrm{proj}}_{r,s}\right]_{r',s'}^{r'',s''} + \left[N^{\mathrm{pseudo}}_{r,s}\right]_{r',s'}^{r'',s''} + \left[N^{\mathrm{off}}_{r,s}\right]_{r',s'}^{r'',s''} \tag{4.20}$$

where we have introduced

$$\begin{aligned} \left[N^{\mathrm{proj}}_{r,s}\right]_{r',s'}^{r'',s''} &= S^{1,p}_{r',s'}F^{1,p}_{r,s}S^{r'',s''}_{1,p} + \sum_{b=1}^{p-1} S^{2,b}_{r',s'}F^{2,b}_{r,s}S^{r'',s''}_{2,b} + S^{2,p}_{r',s'}F^{2,p}_{r,s}S^{r'',s''}_{2,p} \\ \left[N^{\mathrm{pseudo}}_{r,s}\right]_{r',s'}^{r'',s''} &= \sum_{b=1}^{p-1} S^{0,b}_{r',s'}F^{0,b}_{r,s}S^{r'',s''}_{0,b}, \qquad \left[N^{\mathrm{off}}_{r,s}\right]_{r',s'}^{r'',s''} = \sum_{b=1}^{p-1} S^{0,b}_{r',s'}F^{0,b;2,b}_{r,s}S^{r'',s''}_{2,b} \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} F^{1,p}_{r,s} &= \frac{S^{1,p}_{1,p}S^{1,p}_{r,s}}{\left(S^{1,p}_{1,1}\right)^2}, \qquad F^{2,b}_{r,s} = \frac{S^{1,p}_{1,p}S^{0,b}_{r,s}}{S^{1,p}_{1,b}S^{0,b}_{1,1}}, \qquad F^{2,p}_{r,s} = \frac{S^{1,p}_{1,p}S^{2,p}_{r,s}}{S^{1,p}_{1,1}S^{2,p}_{1,1}} \\ F^{0,b}_{r,s} &= \frac{S^{0,b}_{r,s}}{S^{0,b}_{1,1}}, \qquad F^{0,b;2,b}_{r,s} = \frac{S^{1,p}_{1,p}\left(S^{0,b}_{1,1}S^{2,b}_{r,s} - S^{2,b}_{1,1}S^{0,b}_{r,s}\right)}{S^{1,p}_{1,b}\left(S^{0,b}_{1,1}\right)^2}. \end{aligned} \tag{4.22}$$

The Verlinde-like formula (4.18) can also be written in terms of the improper $S$-matrix (2.22). A bit of rewriting thus yields

$$\begin{aligned} [N_{r,s}]_{r',s'}^{r'',s''} = \sum_{\nu=1}^{2} \mathcal{S}^{\nu,p}_{r',s'}\frac{\mathcal{S}^{\nu,p}_{r,s}}{\mathcal{S}^{\nu,p}_{1,1}}\mathcal{S}^{r'',s''}_{\nu,p} + \sum_{b=1}^{p-1}\sum_{\nu=1}^{2}\left(\mathcal{S}^{\nu,b}_{r',s'}\frac{\tilde{\mathcal{S}}^{\nu,b}_{r,s}}{\tilde{\mathcal{S}}^{\nu,b}_{1,1}}\mathcal{S}^{r'',s''}_{\nu,b} + \tilde{\mathcal{S}}^{\nu,b}_{r',s'}\frac{\tilde{\mathcal{S}}^{\nu,b}_{r,s}}{\tilde{\mathcal{S}}^{\nu,b}_{1,1}}\tilde{\mathcal{S}}^{r'',s''}_{\nu,b}\right. \\ \left. + \tilde{\mathcal{S}}^{\nu,b}_{r',s'}\left[\frac{\mathcal{S}^{\nu,b}_{r,s}\tilde{\mathcal{S}}^{\nu,b}_{1,1} - \mathcal{S}^{\nu,b}_{1,1}\tilde{\mathcal{S}}^{\nu,b}_{r,s}}{\left(\tilde{\mathcal{S}}^{\nu,b}_{1,1}\right)^2}\right]\mathcal{S}^{r'',s''}_{\nu,b}\right). \end{aligned} \tag{4.23}$$

A similar but superficially $\tau$-dependent expression was conjectured in [18]. We have managed to prove analytically that their formula is indeed $\tau$-independent and that the manifestly $\tau$-independent part of their formula is equivalent to (4.23). In our notation, the proof of the

| $*$ | $G_{1,1}$ | $G_{2,1}$ | $G_{1,2}$ | $G_{2,2}$ |
|---|---|---|---|---|
| $G_{1,1}$ | $G_{1,1}$ | $G_{2,1}$ | $G_{1,2}$ | $G_{2,2}$ |
| $G_{2,1}$ | $G_{2,1}$ | $G_{1,1}$ | $G_{2,2}$ | $G_{1,2}$ |
| $G_{1,2}$ | $G_{1,2}$ | $G_{2,2}$ | $2G_{1,1}+2G_{2,1}$ | $2G_{1,1}+2G_{2,1}$ |
| $G_{2,2}$ | $G_{2,2}$ | $G_{1,2}$ | $2G_{1,1}+2G_{2,1}$ | $2G_{1,1}+2G_{2,1}$ |

| $*$ | $0$ | $1$ | $-\frac{1}{8}$ | $\frac{3}{8}$ |
|---|---|---|---|---|
| $0$ | $0$ | $1$ | $-\frac{1}{8}$ | $\frac{3}{8}$ |
| $1$ | $1$ | $0$ | $\frac{3}{8}$ | $-\frac{1}{8}$ |
| $-\frac{1}{8}$ | $-\frac{1}{8}$ | $\frac{3}{8}$ | $2(0)+2(1)$ | $2(0)+2(1)$ |
| $\frac{3}{8}$ | $\frac{3}{8}$ | $-\frac{1}{8}$ | $2(0)+2(1)$ | $2(0)+2(1)$ |

**Figure 2.** Cayley tables of the multiplication rules for Grot$[\mathcal{WLM}(1,2)]$. In the second table, the generators $G_{r,s}$ are represented by the corresponding conformal weights.

$\tau$-independence amounts to showing that

$$\sum_{b=1}^{p-1}\sum_{\nu=1}^{2}\frac{\tilde{\mathcal{S}}_{r,s}^{\nu,b}\tilde{\mathcal{S}}_{r',s'}^{\nu,b}\mathcal{S}_{\nu,b}^{r'',s''}}{\tilde{\mathcal{S}}_{1,1}^{\nu,b}} = 0 = \sum_{b=1}^{p-1}\sum_{\nu=1}^{2}\frac{\tilde{\mathcal{S}}_{r,s}^{\nu,b}\tilde{\mathcal{S}}_{r',s'}^{\nu,b}\mathcal{S}_{1,1}^{\nu,b}\tilde{\mathcal{S}}_{\nu,b}^{r'',s''}}{\left(\tilde{\mathcal{S}}_{1,1}^{\nu,b}\right)^2}. \tag{4.24}$$

The Verlinde-like formula (4.23) can be written in a slightly more compact form. From the explicit expressions (2.23) for the matrices $\mathcal{S}$ and $\tilde{\mathcal{S}}$, one can easily see that the inner ratio in the second term of (4.23) has a well-defined limit when $t \to p$, equal to the inner ratio in the first term. So one accounts for the first term if one extends, for the second term, the summation from 1 to $p$. Moreover, the remaining two terms in (4.23) do not change if we extend the summation to $p$, because the inner ratios are again well defined in the limit $t \to p$, but are then multiplied by the entries $\tilde{\mathcal{S}}_{r',s'}^{\nu,p} = 0$. We thus obtain a symmetrical expression, which contains a summation over all $2p$ indices, namely

$$[N_{r,s}]_{r',s'}^{r'',s''} = \sum_{j=1}^{p}\sum_{\nu=1}^{2}\left(\mathcal{S}_{r',s'}^{\nu,j}\frac{\tilde{\mathcal{S}}_{r,s}^{\nu,j}}{\tilde{\mathcal{S}}_{1,1}^{\nu,j}}\mathcal{S}_{\nu,j}^{r'',s''} + \tilde{\mathcal{S}}_{r',s'}^{\nu,j}\frac{\tilde{\mathcal{S}}_{r,s}^{\nu,j}}{\tilde{\mathcal{S}}_{1,1}^{\nu,j}}\tilde{\mathcal{S}}_{\nu,j}^{r'',s''} + \tilde{\mathcal{S}}_{r',s'}^{\nu,j}\left[\frac{\mathcal{S}_{r,s}^{\nu,j}\tilde{\mathcal{S}}_{1,1}^{\nu,j} - \mathcal{S}_{1,1}^{\nu,j}\tilde{\mathcal{S}}_{r,s}^{\nu,j}}{\left(\tilde{\mathcal{S}}_{1,1}^{\nu,j}\right)^2}\right]\mathcal{S}_{\nu,j}^{r'',s''}\right). \tag{4.25}$$

### *4.1. The case* $\mathcal{WLM}(1,2)$

The four-dimensional Grothendieck ring

$$\text{Grot}[\mathcal{WLM}(1,2)] \simeq \mathbb{C}[Y]/(Y^4 - 4Y^2) \tag{4.26}$$

is generated by

$$G_{1,1} \leftrightarrow I, \qquad G_{1,2} \leftrightarrow Y, \qquad G_{2,1} \leftrightarrow \tfrac{1}{2}Y^2 - I, \qquad G_{2,2} \leftrightarrow \tfrac{1}{2}Y^3 - Y. \tag{4.27}$$

The multiplication rules are given in the Cayley tables in figure 2. The fundamental fusion matrix $Y$ is given in (3.14).

There is only one pseudo-character, $\hat{\chi}_{0,1}(q) = \mathrm{i}\tau(\hat{\chi}_{1,1}(q) - \hat{\chi}_{2,1}(q))$, and in the ordered basis $\{\hat{\chi}_{1,1}(q), \hat{\chi}_{2,1}(q), \hat{\chi}_{1,2}(q), \hat{\chi}_{2,2}(q), \hat{\chi}_{0,1}(q)\}$, the proper $S$-matrix is given by

$$S = \begin{pmatrix} 0 & 0 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ -1 & -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ -1 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{4.28}$$

The similarity matrix $Q$ (4.5) and its inverse $Q^{-1}$ (4.7) are given by

$$Q = \begin{pmatrix} \frac{1}{4} & -\frac{1}{2} & 0 & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & 0 & -\frac{1}{4} \\ \frac{1}{2} & 0 & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & -1 & \frac{1}{2} \end{pmatrix}, \qquad Q^{-1} = \begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ -1 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ -1 & -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \tag{4.29}$$

They convert the four Grothendieck matrices $N_{r,s}$, $r, s \in \mathbb{Z}_{1,2}$, simultaneously into the Jordan forms

$$\begin{aligned} &Q^{-1}N_{1,1}Q = \mathrm{diag}(1, 1, 1, 1), \qquad Q^{-1}N_{1,2}Q = \mathrm{diag}\left(2, \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}, -2\right) \\ &Q^{-1}N_{2,1}Q = \mathrm{diag}(1, -1, -1, 1), \qquad Q^{-1}N_{2,2}Q = \mathrm{diag}\left(2, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, -2\right). \end{aligned} \tag{4.30}$$

It is noted that these are not all in Jordan *canonical* form. The Verlinde-like formula (4.18) follows by inverting the decompositions (4.30).

## Acknowledgments

This work is supported by the Australian Research Council and by the Belgian Interuniversity Attraction Poles Program P6/02, through the network NOSY (Nonlinear systems, stochastic processes and statistical mechanics). PR is a Research Associate of the Belgian National Fund for Scientific Research (FNRS). PAP thanks the Asia Pacific Center for Theoretical Physics for hospitality during the early stages of this work.

## References

[1] Behrend R, Pearce P A, Petkova V and Zuber J-B 2000 Boundary conditions in rational conformal field theories *Nucl. Phys.* B **579** 707–73 (arXiv:hep-th/9908036)
[2] Verlinde E 1988 Fusion rules and modular transformations in 2D conformal field theory *Nucl. Phys.* B **300** 360–76
[3] Cardy J L 1986 Operator content of two-dimensional conformally invariant theories *Nucl. Phys.* B **270** 186–204
[4] Cardy J L 1986 Effect of boundary conditions on the operator content of two-dimensional conformally invariant theories *Nucl. Phys.* B **275** 200–18
[5] Cardy J 1989 Boundary conditions, fusion rules and the Verlinde formula *Nucl. Phys.* B **324** 581–96
[6] Flohr M 2003 Bits and pieces in logarithmic conformal field theory *Int. J. Mod. Phys.* A **18** 4497–592 (arXiv:hep-th/0111228)
[7] Gaberdiel M R 2003 An algebraic approach to logarithmic conformal field theory *Int. J. Mod. Phys.* A **18** 4593–638 (arXiv:hep-th/0111260)
[8] Fuchs J, Hwang S, Semikhatov A M and Tipunin I Yu 2004 Nonsemisimple fusion algebras and the Verlinde formula *Commun. Math. Phys.* **247** 713–42 (arXiv:hep-th/0306274)
[9] Pearce P A, Rasmussen J and Zuber J-B 2006 Logarithmic minimal models *J. Stat. Mech.* P11017 (arXiv:hep-th/0607232)
[10] Rasmussen J and Pearce P A 2007 Fusion algebras of logarithmic minimal models *J. Phys. A: Math. Theor.* **40** 13711–33 (arXiv:0707.3189 [hep-th])
[11] Pearce P A, Rasmussen J and Ruelle P 2008 Integrable boundary conditions and $\mathcal{W}$-extended fusion in the logarithmic minimal models $\mathcal{LM}(1, p)$ *J. Phys. A: Math. Theor.* **41** 295201 (arXiv:0803.0785 [hep-th])
[12] Rasmussen J 2009 $\mathcal{W}$-extended logarithmic minimal models *Nucl. Phys.* B **807** 495–533 (arXiv:0805.2991 [hep-th])
[13] Kausch H G 2000 Symplectic fermions *Nucl. Phys.* B **583** 513–41 (arXiv:hep-th/0003029)
[14] Flohr M 1996 On modular invariant partition functions of conformal field theories with logarithmic operators *Int. J. Mod. Phys.* A **11** 4147–72 (arXiv:hep-th/9509166)
[15] Flohr M 1997 On fusion rules in logarithmic conformal field theories *Int. J. Mod. Phys.* A **12** 1943–58 (arXiv:hep-th/9605151)

[16] Flohr M and Gaberdiel M R 2006 Logarithmic torus amplitudes *J. Phys. A: Math. Gen.* **39** 1955–68 (arXiv:hep-th/0509075)

[17] Feigin B L, Gainutdinov A M, Semikhatov A M and Tipunin I Yu 2006 Logarithmic extensions of minimal models: characters and modular transformations *Nucl. Phys.* B **757** 303–43 (arXiv:hep-th/0606196)

[18] Gaberdiel M R and Runkel I 2008 From boundary to bulk in logarithmic CFT *J. Phys. A: Math. Theor.* **41** 075402 (arXiv:0707.0388 [hep-th])

[19] Flohr M and Knuth H 2007 On Verlinde like formulas in $c_{p,1}$ logarithmic conformal field theories arXiv:0705.0545 [math-ph]

[20] Gainutdinov A M and Tipunin I Yu 2007 Radford, Drinfeld, and Cardy boundary states in $(1, p)$ logarithmic conformal field models arXiv:0711.3430 [hep-th]

[21] Rasmussen J 2009 Fusion matrices, generalized Verlinde formulas, and partition functions in $\mathcal{WLM}(1, p)$ arXiv:0908.2014 [hep-th]

[22] Gaberdiel M R and Kausch H G 1996 A rational logarithmic conformal field theory *Phys. Lett.* B **386** 131–7 (arXiv:hep-th/9606050)

[23] Eberle H and Flohr M 2006 Virasoro representations and fusion for general augmented minimal models *J. Phys. A: Math. Gen.* **39** 15245–86 (arXiv:hep-th/0604097)

[24] Rasmussen J 2009 Polynomial fusion rings of $\mathcal{W}$-extended logarithmic minimal models *J. Math. Phys.* **50** 043512 (arXiv:0812.1070 [hep-th])